# Part II

# Multivariate Linear Models

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# 1. Multivariate Regression Models

## 1.1 The Model

T

$$y_i' = x_i'B + u_i'$$

$i = 1, \ldots, n,$ $n$ a . W $\{y_i\}$ a $\{u_i\}$ $\ell$- a , a $\{x_i\}$ $m$- a , a a a a a a a . T $m \times \ell$ a $B$ a ffi . T fi $\ell$- a a $y_i$ a $x_i$, a ffi a $B$. I a a a , . I a , a

$$Y = XB + U$$

a a fi a a, . . ., a a a a a a .

T $\{u_i\}$ a a $(0, \Sigma)$ a a a . T , $\mathbf{E}u_iu_j' = \Sigma$ $i = j$ a 0 . M , a $(Z) =$ a ( $Z$),

$$\text{a } (U) = I_n \otimes \Sigma$$

a .

## 1.2 Multivariate Least Squares

S $\sum_{p,q} z_{pq}^2 =$ $Z'Z$ a a $Z = (z_{pq})$, a a (LS) a $\widehat{B}$ $B$ fi

$$\widehat{B} = \text{a}_{B} \quad (Y - XB)'(Y - XB) \tag{1}$$

I $q(B) =$ $(Y - XB)'(Y - XB),$ a

$$dq(B) = -2 \quad (X'Y - X'XB)'\,dB$$

a FOC $dq(B) = 0$

$$\widehat{B} = (X'X)^{-1}X'Y$$

S a a a a , $\widehat{Y} = X\widehat{B}$ a $\widehat{U} = Y - X\widehat{B}$. T

$$\widehat{Y} = P_XY \quad \text{a} \quad \widehat{U} = (I - P_X)Y$$

$P_X = X(X'X)^{-1}X'$, . ., a $\mathcal{R}(X)$ $X$, a a a .

I a a $\widehat{B}$ , a (1),

$$q(B) = \quad (Y - X\widehat{B})'(Y - X\widehat{B}) + \quad (\widehat{B} - B)'X'X(\widehat{B} - B) \tag{2}$$

N a LS a $\widehat{B}$ LS a , $\ell$- a a a . T a , $k$- $\widehat{B}$ LS a , $y_k$ $X$, $y_k$ $k$- , $Y$.

G LS a $\widehat{B}$, a a $\Sigma$

$$
\begin{aligned}
\widehat{\Sigma} &= \frac{1}{n}\sum_{t=1}^{n} \hat{u}_i \hat{u}_i' \quad \left(= \frac{1}{n}\widehat{U}'\widehat{U}\right) \\
&= \frac{1}{n} Y'(I - P_X)Y \quad \left(= \frac{1}{n} U'(I - P_X)U\right)
\end{aligned}
$$

## 1.3 ML Estimation

U a , $U \sim \mathbf{N}(0, I_n \otimes \Sigma)$ a

$$
p(U) = (2\pi)^{-\frac{n\ell}{2}} ( \quad \Sigma)^{-\frac{n}{2}} \quad \left(-\frac{1}{2}\Sigma^{-1}U'U\right)
$$

$(\cdot) :=$ ( $(\cdot)$). T a a , ,

$$
\begin{aligned}
(I_n \otimes \Sigma) &= ( \quad \Sigma)^n \\
( \quad U)'(I_n \otimes \Sigma^{-1})( \quad U) &= \quad \Sigma^{-1}U'U
\end{aligned}
$$

T , $Y$ a a $U$ $Y - XB$, Ja a , a , a 1.

I a , , ,

$$
\ell(B, \Sigma) = -\frac{n}{2} ( \quad \Sigma) - \frac{1}{2} \quad \Sigma^{-1}(Y - XB)'(Y - XB)
$$

T a ff a $\ell(B, \Sigma)$ $B$ a $\Sigma$ fi

$$
\begin{aligned}
\Sigma^{-1}(X'Y - X'XB)'\, dB &= 0 \\
\Sigma^{-1}\left(I - \frac{1}{n}(Y - XB)'(Y - XB)\Sigma^{-1}\right) d\Sigma &= 0
\end{aligned}
$$

T a FOC a ,

(a) $d ( \quad \Sigma) = \quad \Sigma^{-1}\, d\Sigma$
( ) $d\Sigma^{-1} = -\Sigma^{-1}\, d\Sigma\, \Sigma^{-1}$

F (a), a $(\partial/\partial\sigma_{pq})$ $\Sigma = c_{pq}$, $\Sigma = (\sigma_{pq})$ a $c_{pq}$ , a , $\sigma_{pq}$ a a $\Sigma = (c_{qp})$. T a , a $\Sigma/ \quad \Sigma = \Sigma^{-1}$. Pa ( ) a a a a ff a $\Sigma\Sigma^{-1} = I$.

I a a FOC' LS a $\widehat{B}$ a $\widehat{\Sigma}$ a a . I a a a , $Y$. T a $\widehat{B}$ a (2). M , a a P 1 ( $A = \widehat{\Sigma}^{\frac{1}{2}}\Sigma^{-1}\widehat{\Sigma}^{\frac{1}{2}}$) a

$$
\ell(\widehat{B}, \Sigma) = -\frac{n}{2} ( \quad \Sigma) - \frac{n}{2} \quad \Sigma^{-1}\widehat{\Sigma}
$$

a $\Sigma = \widehat{\Sigma}$. C , LS a a ML -
a a a a , a a a
, .

## 1.4 Statistical Properties of the Estimators

W a a a a $\widehat{B}$ a $\widehat{\Sigma}$.

**Theorem 1** *We have*

(a) $\mathbf{E}\,(\widehat{B}) = B$ a a $(\widehat{B}) = (X'X)^{-1} \otimes \Sigma$

( ) $\mathbf{E}\,(\widehat{\Sigma}) = \dfrac{n-m}{n}\,\Sigma$

**Proof** Pa (a) a a

$$\widehat{B} = B + (X'X)^{-1}X'U \quad \text{a} \quad \widehat{B} = \quad B + ((X'X)^{-1}X' \otimes I_\ell) \quad U$$

F a ( ),

$$\widehat{\Sigma} = \frac{1}{n}U'(I - P_X)U = \frac{1}{n}V'V$$

$V := H'U$, $H$ a $n \times (n{-}m)$ a a $I - P_X = HH'$ a $H'H = I_{n-m}$.
N , $V = (H' \otimes I)$ $U$ a a $V = I_{n-m} \otimes \Sigma$, . ., $(n{-}m)$ $v_i$' ,
$V$ a , a a a a a a $\Sigma$. T a

$$\mathbf{E}\,V'V = \mathbf{E}\left(\sum_{i=1}^{n-m} v_i v_i'\right) = (n{-}m)\,\Sigma$$

a . ■

T $\widehat{B}$ a , $\widehat{\Sigma}$ . H , a a a a a
$\Sigma$ a

$$\tilde{\Sigma} = \frac{1}{n{-}m}\,Y'(I - P_X)Y$$

F a , $\widehat{B}$ a a
a a a a (a) T 1, $\widehat{B}$ a a a a , $Y$. T
a a , $\widehat{\Sigma}$,

**Definition 1** *Let $z_i \sim$ i.i.d. $\mathbf{N}(0, \Sigma_p)$. Then*

$$\sum_{i=1}^{n} z_i z_i' \sim \mathcal{W}_p\,(n, \Sigma)$$

*i.e., Wishart distribution with $n$ degrees of freedom and covariance matrix $\Sigma$. The $p$ is the dimensionality parameter.*

C a , W a a a a - a .
F a ( ) T 1, a a a ,
$\widehat{\Sigma}$ W a , a a :

**Theorem 2** *Under normality, we have*
(a) $\widehat{B} \sim \mathbf{N}\left(B, (X'X)^{-1} \otimes \Sigma\right)$
( ) $n\,\widehat{\Sigma} \sim \mathcal{W}_\ell\,(n-m, \Sigma)$

T a a a a a a

a a a . U a a a

$$\text{(a)} \quad \frac{X'X}{n} = \sum \frac{x_i x_i'}{n} \to_p M > 0 \quad \text{a} \qquad (\ ) \quad \frac{X'U}{n} = \sum \frac{x_i u_i'}{n} \to_p 0$$

$\widehat{B}$ . M , ,

$$(\ ) \quad \frac{X'U}{\sqrt{n}} = \sum \frac{x_i u_i'}{\sqrt{n}} \;\to_d\; \mathbf{N}(0, M \otimes \Sigma)$$

a fi , a

$$\sqrt{n}\,(\widehat{B} - B) \;\to_d\; \mathbf{N}\left(0, M^{-1} \otimes \Sigma\right) \tag{3}$$

$(\widehat{B}-B) = \left((X'X)^{-1} \otimes I_\ell\right)$ $X'U$. T $\widehat{\Sigma}$ a a a , a a $\hat{\sigma}^2$.

## 1.5 Hypothesis Testing

T a a a a a

a a . T , fl a T a a ffi a $B$ a a a

$$R \quad B = r$$

$R$ a $r$ a $q \times m\ell$ a $q \times 1$, , a fi a a a . U a , a Wa a $W$

$$W = (R \quad \widehat{B} - r)' \left(R((X'X)^{-1} \otimes \widehat{\Sigma})R'\right)^{-1} (R \quad \widehat{B} - r)$$

T $W$ a a a
(3) a

$$W \;\to_d\; \chi^2_q$$

T $\widehat{\Sigma}$ a a a a $\Sigma$ a
- a .

T fi a $W$ a , a a a a . T , a .

## 1.6 Exercises

1. S :

(a) $(ABC) = (C' \otimes A)( B)$.

( ) L $x =$ $X$ a $y =$ $Y$. T

$$\frac{\partial^2}{\partial x \, \partial y'} \quad X'AYB' = A \otimes B$$

2. S a a

$$\ell(\widehat{B}, \Sigma) = -\frac{n}{2} \quad ( \quad \Sigma) - \frac{n}{2} \quad \Sigma^{-1}\widehat{\Sigma},$$

S 1.3 a $\Sigma = \widehat{\Sigma}$.

3. L $\hat{B}$ GLS a $B$ a a

$$Y = XB + U$$

a $(U) = \Sigma_1 \otimes \Sigma_2$, fi a $\Sigma_1$ a $\Sigma_2$. S a $\hat{B}$

$$\Sigma_2^{-1}(Y - XB)'\Sigma_1^{-1}(Y - XB)$$

a

$$\hat{B} = (X'\Sigma_1^{-1}X)^{-1}X'\Sigma_1^{-1}Y$$

4. C a a a

$$y_i' = x_i'B + u_i'$$

$y_i' = (y_{1i}, y_{2i}), x_i' = (x_{1i}, x_{2i}), u_i' = (u_{1i}, u_{2i}),$

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

a $(u_i)$ a a 0 a a a

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

A :

(a) L $\beta_{21} = 0$ . C a a a a a a $\beta_{11}$, $\beta_{12}$ a $\beta_{22}$.

( ) L $\beta_{21} = 0$ *and* $\sigma_{12} = \sigma_{21} = 0$ . H a (a) a ?

( ) A $\Sigma = \sigma^2 I$. C $\beta_{11} + \beta_{12} = 1$. E a

Wa a ff a : OLS a , $\beta_{11}$ a $\beta_{12}$, a
a a a .

5. C

(A) $y_{ij} = x'_{ij}\beta + \varepsilon_{ij}$

(B) $y_{ij} = x'_{ij}\beta_j + \varepsilon_{ij}$

$i = 1, \ldots, n$ a $j = 1, 2$. L $\varepsilon_i = (\varepsilon_{1i}, \varepsilon_{2i})'$ a a a

$$\Sigma_1 = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}.$$

A :

(a) E a fi a GLS a $\beta$ (A)
a a a $\Sigma_1$.

( ) F GLS a $\beta_j, j = 1, 2,$ (B) a a
a $\Sigma_1$ $\Sigma_2$. C a GLS a OLS a a a . W a
$x_{i1} = x_{i2}$ (B)?

# 2. Seemingly Unrelated Regressions

## 2.1 The Model

T a (SUR) a

$$y_k = X_k\beta_k + u_k$$

$k = 1, \ldots, \ell,$ $y_k, X_k, \beta_k$ a $u_k$ a fi a a a a
a a a $k$ $k$- a , a
a $n$ a . T a a a , a
$\ell$- a a

$$\begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix} = \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_\ell \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_\ell \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$$

a

$$y = X\beta + u$$

a . W $\mathbf{E}\, u_p u_q' = \sigma_{pq} I$ a $\Sigma = (\sigma_{pq})$. T a a $u = \Sigma \otimes I_n$.
N a a a a $U$ a $u = \overline{\quad} U$ a a $U =$ a ( $U) = I_n \otimes \Sigma$.

T a a a a
SUR $X_1 = \cdots = X_\ell$, . ., SUR a . I

$X_0$, a a a . SUR

$$\text{vec}\, Y = (I \otimes X_0)\,\text{vec}\, B + \text{vec}\, U$$

a fi $y = \text{vec}\,Y, X = I \otimes X_0, \beta = \text{vec}\,B$ a $u = \text{vec}\,U$.

## 2.2 Estimation

T SUR a ffi a GLS,

$$\hat{\beta} = \left(X'(\Sigma^{-1} \otimes I)X\right)^{-1} X'(\Sigma^{-1} \otimes I)y \tag{4}$$

T a SUR a , GLS a . W $\Sigma$ , a a a , a a GLS a . T a $\Sigma$, a a $\hat{\beta}_k$ $\beta_k$ ( OLS a , a ) $\hat{u}_k = y_k - X_k\hat{\beta}_k$ $k = 1, \ldots, \ell$. U $\{\hat{u}_k\}$, a a $\hat{\sigma}_{pq} = \hat{u}_p'\hat{u}_q/n$ $p, q = 1, \ldots, \ell$ a a a $\widehat{\Sigma} = (\hat{\sigma}_{pq})$.

U a , $u$ a a a $U$ fi $u$ $\text{vec}\,U = u$. M , $y$ a a a a $U$ a $k$- $U$ $y_k - X_k\beta_k$. T SUR a fi (4) , ML a $\beta$ a . G ML a $\beta$, ML a $\Sigma$ a a a a a . T a , fi $\widehat{U}$ $\hat{u}$, ML ( SUR, a ) a , a fi $U$ $u$,

$$\widehat{\Sigma} = \frac{1}{n}\widehat{U}'\widehat{U}$$

T ML a $\beta$ a $\Sigma$ a a a a GLS.

I a OLS SUR a a (SELS). I $\tilde{\beta}$ OLS a $\beta$ SUR ,

$$\begin{aligned} \tilde{\beta} &= (X'X)^{-1}X'y \\ &= \begin{pmatrix} (X_1'X_1)^{-1}X_1'y_1 \\ \vdots \\ (X_\ell'X_\ell)^{-1}X_\ell'y_\ell \end{pmatrix} \end{aligned}$$

OLS a a a a a .
F SUR , GLS ffi a OLS a a . T a SUR a ffi a SELS, a . H a a a SUR: W a ffi a (a a ).

## 2.3 Equivalence of SUR and SELS

T a a SUR a SELS ( OLS). F , a $\sigma_{pq} = 0$ a $p \neq q$, . ., a a . S , a a a a a , a a a , a $X$ $I \otimes X_0$. I a a a SUR a SELS a .

T SUR SELS:

**Theorem 3** *The* SUR *and* SELS *are identical if and only if*

$$\sigma_{pq} = 0 \qquad \mathcal{R}(X_p) = \mathcal{R}(X_q)$$

*for all* $p \neq q$.

**Proof** B K a' , a a a $\mathcal{R}((\Sigma \otimes I)X) = \mathcal{R}(X)$. S a $((\Sigma \otimes I)X) =$ a $(X)$, ffi a

$$\mathcal{R}((\Sigma \otimes I)X) \subset \mathcal{R}(X)$$

a a a $T$ a

$$(\Sigma \otimes I)X = XT$$

I $T = (T_{pq})$ a $T_{pq}$,

$$\sigma_{pq} X_q = X_p T_{pq}$$

a $p, q$, a

$$\sigma_{pq} = 0 \qquad \mathcal{R}(X_p) = \mathcal{R}(X_q)$$

a $p \neq q$, a a . ■

## 2.4 Exercises

1. C a SUR

$$\begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix} = \begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_\ell \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_\ell \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$$

$y_k$ $(n \times 1)$, $X_k$ $(n \times m)$, $\beta_k$ $(m \times 1)$ a $u_k$ $(n \times 1)$ a $k = 1, \ldots, \ell$. W a a $\mathbf{E}u_k = 0$ a $\mathbf{E}u_j u_k' = \sigma_{jk} I_n$ a $j, k = 1, \ldots, \ell$. W a a $X_k$ - a a $k = 1, \ldots, \ell$ . W a a

$$y = X\beta + u,$$

$y$, $X$, $\beta$ a $u$ a fi a a a . A , :

(a) D a , SUR a , $\beta = (\beta_1', \ldots, \beta_\ell)'$.

( ) S a SUR a , $\beta$ a a a a , a - - a OLS a $\sigma_{jk} = 0$ , a $j \neq k$.

( ) S a SUR a , $\beta$ a a a a , a - - a OLS a $X_1 = \ldots = X_\ell$.

2. C a a SUR

$$\begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix} = \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_\ell \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_\ell \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_\ell \end{pmatrix}$$

a fi Q 1 a . W , $\mathbf{E}\, u_p u_q' = \sigma_{pq} I$ a $\Sigma = (\sigma_{pq})$. A , , $\beta = (\beta_1, \ldots, \beta_\ell)'$. E , a a a ML a , a a $\beta$ a $\Sigma$ a . S , a a , a a a ML a a . R a a , $n$- a a a a $\mu$ a a a $\Omega$

$$p(x) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} ( \quad \Omega)^{-\frac{1}{2}} \quad \left(-\frac{1}{2}(x-\mu)'\Omega^{-1}(x-\mu)\right)$$

3. C a

$$\begin{aligned} y_{1i} &= \alpha' x_{1i} + \varepsilon_{1i} \\ y_{2i} &= \alpha' x_{2i} + \beta' x_{3i} + \varepsilon_{2i} \end{aligned}$$

$\{\varepsilon_i\}$, $\varepsilon_i = (\varepsilon_{1i}, \varepsilon_{2i})'$, a a $(\varepsilon_i) = \sigma^2 I$. L $\hat{\alpha}_1$ a $\hat{\alpha}_2$ OLS a a fi a , a $\hat{\alpha}$ OLS a , $\alpha$. A , :

(a) O a $\hat{\alpha}$, a a a a

$$\hat{\alpha} = M_* \hat{\alpha}_1 + (I - M_*)\hat{\alpha}_2$$

$$M_* = (X_1' X_1 + X_2'(I - P_{X_3})X_2)^{-1} X_1' X_1$$

( ) S a $\hat{\alpha}$ , a $(\hat{\alpha}_1 - \hat{\alpha}_2)$. U , a , a $\hat{\alpha}$ a a a a ( fi ) a a a , ,

$$\tilde{\alpha} = M\hat{\alpha}_1 + (I - M)\hat{\alpha}_2$$

(N a $\tilde{\alpha} = \hat{\alpha} + (M - M_*)(\hat{\alpha}_1 - \hat{\alpha}_2)$).

# 3. Fixed Effects Models

## 3.1 The Model

T

$$y_{it} = \pi + \mu_i + \nu_t + x'_{it}\beta + \varepsilon_{it}$$

$i = 1, \ldots, a$, $t = 1, \ldots, b$. T $i$ a $t$ a a , a $\mu_i$ a $\nu_t$ a a ff , . I a a a . T $\{\varepsilon_{it}\}$ a a a $(0, \sigma_\varepsilon^2)$. W $n = ab$ a .

I a , a a

$$y = \iota_n \pi + (I_a \otimes \iota_b)\mu + (\iota_a \otimes I_b)\nu + X\beta + \varepsilon$$

$y = (y_{11}, y_{12}, \ldots, y_{1b}, \ldots, y_{a1}, y_{a2}, \ldots, y_{ab})'$, a $X$ a $\varepsilon$ a fi a, $\{x_{it}\}$ a $\{\varepsilon_{it}\}$. M , $\mu = (\mu_1, \ldots, \mu_a)'$ a $\nu = (\nu_1, \ldots, \nu_b)'$. T a $I$ a $\iota$ a . I , "_" a a a . I a a , a a, . ., a a $i$ a $t$. F a , $\bar{y}_i$, $\bar{y}_t$ a $\bar{y}$ , , a a , $y_{it}$' $t$, $i$ a $i$ a $t$.

L a ff $\mu$ a $\nu$ fi a a . T a a a a a , a a a *dummy variables model*. T a a $\mu$ a $\nu$ a , , fi $\iota_n$, $I_a \otimes \iota_b$ a $\iota_a \otimes I_b$ a a . T , , a a . T a

$$\sum_{i=1}^{a} \mu_i = \sum_{t=1}^{b} \nu_t = 0 \tag{5}$$

a $\mu$ a $\nu$ ff fi a .

## 3.2 Estimation

W fi

$$I - \frac{\iota_a \iota'_a}{a} = H_a H'_a \quad \text{a} \quad I - \frac{\iota_b \iota'_b}{b} = H_b H'_b$$

$H_a$ a $H_b$ a a a a $H'_a H_a = I_{a-1}$ a $H'_b H_b = I_{b-1}$, a

$$H'_a \mu = \mu^* \quad \text{a} \quad H'_b \nu = \nu^*$$

N a $H_a H'_a \mu = \mu$ a $H_b H'_b \nu = \nu$ (5). W a a a $\mu^*$ a $\nu^*$ a

$$y = \iota_n \pi + (H_a \otimes \iota_b)\mu^* + (\iota_a \otimes H_b)\nu^* + X\beta + \varepsilon$$

T a a , a $\iota_n$, $H_a \otimes \iota_b$ a $\iota_a \otimes H_b$ a
a a a a . T $P$
a . M , a

$$
\begin{aligned}
P &= \frac{\iota_n \iota_n'}{n} + \left( I_a - \frac{\iota_a \iota_a'}{a} \right) \otimes \frac{\iota_b \iota_b'}{b} + \frac{\iota_a \iota_a'}{a} \otimes \left( I_b - \frac{\iota_b \iota_b'}{b} \right) \\
&= I_a \otimes \frac{\iota_b \iota_b'}{b} + \frac{\iota_a \iota_a'}{a} \otimes I_b - \frac{\iota_n \iota_n'}{n}
\end{aligned}
$$

L $Q = I - P$ a OLS a $\hat{\beta}$ $\beta$ a

$$
\hat{\beta} = (X'QX)^{-1} X'Qy \tag{6}
$$

N a

$$
I_a \otimes \frac{\iota_b \iota_b'}{b} \ , \quad \frac{\iota_a \iota_a'}{a} \otimes I_b \ , \quad \frac{\iota_n \iota_n'}{n}
$$

a a a a a $t$, $i$ a $i$ a $t$, . I a
a a $\hat{\beta}$ OLS a $\beta$

$$
y_{it} - \bar{y}_i - \bar{y}_t + \bar{y} = (x_{it} - \bar{x}_i - \bar{x}_t + \bar{x})'\beta + e_{it}
$$

T OLS a $\pi$, $\mu^*$ a $\nu^*$ a a

$$
y - X\hat{\beta} = \iota_n \pi + (H_a \otimes \iota_b)\mu^* + (\iota_a \otimes H_b)\nu^* + \varepsilon
$$

S a a a , a a a
a a . F a $\mu$ a $\nu$, a

$$
\mu = H_a \mu^* \quad \text{a} \quad \nu = H_b \nu^*
$$

W a OLS a $\pi$, $\mu$ a $\nu$ a

$$
\hat{\pi} = \bar{y} - \bar{x}'\hat{\beta} \ , \quad \hat{\mu}_i = (\bar{y}_i - \bar{y}) - (\bar{x}_i - \bar{x})'\hat{\beta} \ , \quad \hat{\nu}_t = (\bar{y}_t - \bar{y}) - (\bar{x}_t - \bar{x})'\hat{\beta}
$$

### 3.3 Exercises

1. C

$$
y_{it} = \mu_i + x_{it}'\beta_i + \varepsilon_{it}
$$

$i = 1, \ldots, a$ a $t = 1, \ldots, b$. A $\{x_{it}\}$ a $m$- a a a , a $\{\varepsilon_{it}\}$ a . . . $\mathbf{N}(0,\sigma^2)$. C F- a $\beta_1 = \cdots = \beta_a$.

2. C a a

$$
\begin{aligned}
y_{ij} &= \pi + \mu_j + x_{ij}'\beta + \varepsilon_{ij}, \\
y_{ij} &= \nu_j + x_{ij}'\beta + \varepsilon_{ij}
\end{aligned}
$$

$\sum_j \mu_j = 0$.

(a) S a OLS a $\beta$ a a.
( ) L $\hat{\pi}$, $\hat{\mu}_j$ a $\hat{\nu}_j$ OLS a $\pi$, $\mu_j$ a $\nu_j$. R a $\hat{\pi}$, $\hat{\mu}_j$ a $\hat{\nu}_j$.

3. C a

$$y_{it} = \pi + \mu_t + \eta_i + \varepsilon_{it}$$

$i = 1, \ldots, a$ a $t = 1, \ldots, b$, $(\eta_i)$ a $(\varepsilon_{it})$ a a a, a a $(\eta_i) = \sigma_\eta^2$ a a $(\varepsilon_{it}) = \sigma_\varepsilon^2$. A :
(a) O a GLS a $\pi$ a $\mu_t$' $t = 1, \ldots, b$.
( ) C a OLS a GLS a .

4. C fi ff

$$y_{it} = \pi + \mu_i + \nu_t + x_{it}'\beta + \varepsilon_{it}$$

$i = 1, \ldots, a$ a $t = 1, \ldots, b$, $(\varepsilon_{it})$ a a $(0, \sigma_\varepsilon^2)$.
(a) D fia a a $(\mu_i)$ a $(\nu_t)$ a a ff .
( ) P $(\mu_i)$ a $(\nu_t)$ a a a . D fia a a a a .
( ) O a OLS a a a $\pi$, $(\mu_i)$, $(\nu_t)$ a $\beta$ a .

# 4. Random Effects Models

## 4.1 The Model

W

$$y_{it} = \pi + x_t'\beta + \mu_i + \nu_t + \varepsilon_{it}$$

a . H , $\mu$ a $\nu$ a a ff a , a . A $\{\mu_i\}$ a $\{\nu_t\}$ a a $(0, \sigma_\mu^2)$ a $(0, \sigma_\nu^2)$, a a a $\{\varepsilon_{it}\}$. T a a $y_{it}$ a a , . ., $\sigma_\mu^2 + \sigma_\nu^2 + \sigma_\varepsilon^2$, a a a a *error components model.* W a a

$$y = \iota_n \pi + X\beta + u$$

$$u = (I_a \otimes \iota_b)\mu + (\iota_a \otimes I_b)\nu + \varepsilon$$

T a a $\Sigma$ $u$

$$\Sigma = \sigma_\mu^2 (I_a \otimes \iota_b \iota_b') + \sigma_\nu^2 (\iota_a \iota_a' \otimes I_b) + \sigma_\varepsilon^2 I_n$$

### 4.2 Estimation of Regression Coefficients

T a a GLS a , $\beta$ a $\pi$, fi a
a a a a a $I$:

$$\begin{aligned} P_1 &= \left(I_a - \frac{\iota_a\iota_a'}{a}\right) \otimes \left(I_b - \frac{\iota_b\iota_b'}{b}\right) \\ P_2 &= \left(I_a - \frac{\iota_a\iota_a'}{a}\right) \otimes \frac{\iota_b\iota_b'}{b} \\ P_3 &= \frac{\iota_a\iota_a'}{a} \otimes \left(I_b - \frac{\iota_b\iota_b'}{b}\right) \\ P_4 &= \frac{\iota_a\iota_a'}{a} \otimes \frac{\iota_b\iota_b'}{b} \end{aligned}$$

T a

$$\Sigma = \sigma_\varepsilon^2 P_1 + (b\sigma_\mu^2 + \sigma_\varepsilon^2)P_2 + (a\sigma_\nu^2 + \sigma_\varepsilon^2)P_3 + (b\sigma_\mu^2 + a\sigma_\nu^2 + \sigma_\varepsilon^2)P_4$$

W $\Sigma = \sigma_\varepsilon^2 \Sigma_0$ a $\Sigma_0^{-1} = \sum_{k=1}^4 \lambda_k P_k$, $\lambda_k$' a fi a a .

C a a ,

$$\hat{\beta} = (X'QX)^{-1}X'Qy \tag{7}$$

$Q$

$$\begin{aligned} Q &= \Sigma_0^{-1} - \lambda_4 \frac{\iota_n\iota_n'}{n} \\ &= I_n - \varphi_1\left(I_a \otimes \frac{\iota_b\iota_b'}{b}\right) - \varphi_2\left(\frac{\iota_a\iota_a'}{a} \otimes I_b\right) + \varphi_3 \frac{\iota_n\iota_n}{n} \end{aligned}$$

$\varphi_1 = \lambda_1 - \lambda_2,\ \varphi_2 = \lambda_1 - \lambda_3,\ \varphi_3 = \lambda_1 - \lambda_2 - \lambda_3$, a a

$$\varphi_1 = \frac{b\sigma_\mu^2}{b\sigma_\mu^2 + \sigma_\varepsilon^2},\ \varphi_2 = \frac{a\sigma_\nu^2}{a\sigma_\nu^2 + \sigma_\varepsilon^2},\ \varphi_3 = \frac{ab\sigma_\mu^2\sigma_\nu^2 - \sigma_\varepsilon^4}{(a\sigma_\nu^2 + \sigma_\varepsilon^2)(b\sigma_\mu^2 + \sigma_\varepsilon^2)}$$

M , GLS a $\pi$ a a GLS $y - X\hat{\beta} = \iota_n\pi + u$,
a OLS a $\iota_n$ a $\Sigma$. I

$$\hat{\pi} = \bar{y} - \bar{x}'\hat{\beta} \tag{8}$$

A $a, b \to \infty$, $\varphi_1, \varphi_2, \varphi_3 \to 1$ a $Q$ fi
fi ff a (6) . I a , fi ff a
a ff a a $\beta$ a $\pi$ a a .

## 4.3 Estimation of Error Components

T a a, a a . W ,

$$\sigma_1^2 = \sigma_\varepsilon^2 \ , \ \sigma_2^2 = b\sigma_\mu^2 + \sigma_\varepsilon^2 \ , \ \sigma_3^2 = a\sigma_\nu^2 + \sigma_\varepsilon^2$$

a fi

$$\hat\sigma_k^2 = \frac{(y - X\beta)' P_k (y - X\beta)}{d_k} \tag{9}$$

$k = 1, 2, 3,$ $d_k$' a $P_k$' a $d_1 = (a-1)(b-1), d_2 = a-1$ a $d_3 = b-1$. I a a $\mathbf{E}\hat\sigma_k^2 = \sigma_k^2$. N a $\pi$- a a (9) a $P_k$' a a a .

F a a a $\hat\sigma_k^2$' (9), a a $\beta$ a . T OLS a . A a a a , a a $\beta$ a $\hat\sigma_k^2$' a

$$\begin{aligned} y_{it} - \bar y_i - \bar y_t + \bar y &= (x_{it} - \bar x_i - \bar x_t + \bar x)'\beta + e_{it} && (10) \\ \bar y_i - \bar y &= (\bar x_i - \bar x)'\beta + e_i && (11) \\ \bar y_t - \bar y &= (\bar x_t - \bar x)'\beta + e_t && (12) \end{aligned}$$

a $(y - X\beta)' P_k (y - X\beta)$' a RSS' . I a a $\beta$ (10) a a RSS' (11) a (12). T a , (11) a (12) a a $a$ a $b$ a , , a fi a $n$ a .

I ML a a a , , fi a

$$\Sigma = \sigma_1^{2d_1} \sigma_2^{2d_2} \sigma_3^{2d_3} (\sigma_2^2 + \sigma_3^2 - \sigma_1^2)$$

a fi a . M , ML a $\hat\beta$ a $\hat\pi$ $\beta$ a $\pi$, a a GLS a (7) a (8), a

$$(y - \iota_n \hat\pi - X\hat\beta)' \Sigma^{-1} (y - \iota_n \hat\pi - X\hat\beta) = \sum_{k=1}^{3} \frac{(y - X\hat\beta)' P_k (y - X\hat\beta)}{\sigma_k^2}$$

N a a $P_4(y - \iota_n\hat\pi - X\hat\beta) = \bar y - \hat\pi - \bar x'\hat\beta = 0$. T $\sigma_2^2 + \sigma_3^2 - \sigma_1^2$ $\Sigma$ a a ML a a a . W , a fi (9) a ML a . F , a a . N a a a $\sigma_\mu^2$ a $\sigma_\nu^2$ a a $\sigma_k^2$' a a a a . T a .

## 4.4 Exercises

1\. C a ff

$$y_{it} = \pi + x_{it}'\beta + u_{it}$$

$i = 1, \ldots, a$ a $t = 1, \ldots, b$, fi a

$$u_{it} = \mu_i + \nu_t + \varepsilon_{it}$$

a a a ff , $(\mu_i)$ a $(\nu_t)$, a a a $(0, \sigma_\mu^2)$ a $(0, \sigma_\nu^2)$, a a a $(\varepsilon_{it})$. F a a $(\varepsilon_{it})$ a a $(0, \sigma_\varepsilon^2)$, a a .

(a) O a GLS a $\pi$ a $\beta$, a a ff a .

( ) C a a ff a $\beta$ a (a) fi ff a $\beta$ a Q 1( ) a . W a ?

( ) E a a a a a GLS a $\beta$.

2\. S a ff a $\nu_t = 0$. D fi *within*- a *between*- a $\hat{\beta}_w$ a $\hat{\beta}_b$ $\beta$ LS a

$$\begin{aligned} y_{it} - \bar{y}_i &= (x_{it} - \bar{x}_i)'\beta + \varepsilon_{it} \\ \bar{y}_i - \bar{y} &= (\bar{x}_i - \bar{x})'\beta + \varepsilon_i \end{aligned}$$

. S a

(a) T fi ff - a $\hat{\beta}_w$ $\beta$.

( ) S $(A_1 + A_2)^{-1}(B_1 + B_2) = ((A_1 + A_2)^{-1}A_1)A_1^{-1}B_1 + ((A_1 + A_2)^{-1}A_2)A_2^{-1}B_2$ a a ff a $\hat{\beta}$ $\beta$ a a $\hat{\beta} = \Delta\hat{\beta}_w + (I - \Delta)\hat{\beta}_b$ $\Delta$.

3\. C

$$\begin{aligned} y_{it} &= \pi + \mu_i + x_{it}'\beta + \varepsilon_{it} \\ \mu_i &= \bar{x}_i'\alpha + \eta_i \end{aligned}$$

$i = 1, \ldots, a$ a $t = 1, \ldots, b$, $\{\varepsilon_{it}\}$ a $\{\eta_i\}$ a a $(0, \sigma_\varepsilon^2)$ a $(0, \sigma_\eta^2)$, . W a

$$y_{it} = \pi + \bar{x}_i'\alpha + x_{it}'\beta + u_{it}$$

$u_{it} = \eta_i + \varepsilon_{it}$. S a GLS a $\hat{\pi}, \hat{\alpha}$ a $\hat{\beta}$ a a $\pi, \alpha$ a $\beta$ a

$$\hat{\pi} = \bar{y} - \bar{x}'\hat{\beta}_b\ , \quad \hat{\alpha} = \hat{\beta}_b - \hat{\beta}_w\ , \quad \hat{\beta} = \hat{\beta}_w$$

$\hat{\beta}_w$ a $\hat{\beta}_b$ a a fi P 2.

4\. D ML a a a . E a .